A STOCHASTIC STUDY OF A FOUR SPECIES SYN-ECOSYSTEM WITH BIO-ECONOMIC YIELDING OF BOTH VICTIM & KILLER

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ABSTRACT

This paper investigates the stochastic stability of a four species syn-ecosystem with bionomic harvesting of both victim (prey) and killer (predator) species. The necessary conditions for the existence of positive solutions are obtained at different steady states. Moreover we analyzed the local and global stability of deterministic model. The likelihood of survival of bio-economic balance is being conversed. We investigated the inhabitant intensities of fluctuations around the interior equilibrium due to noise. We also discussed the physical significance of the population variances on stability. Finally, we carried out the given numerical simulations to visualize the analytical results using Matlab.

KEYWORDS: *Commensal, steady state, Routh-Hurwitz criteria, global stability, bionomic harvesting, stochastic perturbation, Fourier transforms.*

I. INTRODUCTION

There has been a rising curiosity in the study of harvesting and randomly fluctuating driving forces in a prey-predator-host-commensal system. It is observed in nature that species do not exist in solitary. While species are in the presence of harvesting and randomly fluctuating driving forces, they fight for food, space and are predated by other species. Consequently it is more of natural significance to consider the effect of interaction between species when we study the dynamical behaviour of conventional syn-ecosystem models. So a suitable mathematical model is required to study the effect of harvesting and noise on the interacting species.

Moreover ecology is the study of the inter-relationship between creatures and their surroundings. As it is usual that when there are two or more species live in a common territory, they interact with each other in dissimilar ways. Mathematical modeling has been playing an important role for the last half a century in explaining several phenomena that are concerned with individuals or groups in nature. Lotka [1] and Volterra [2] established theoretical ecology momentously and opened new epochs in the filed of life and biological sciences. The Ecological interactions can be broadly classified as Ammensalism, Commensalism, Competition, Mutualism, Neutralism, Predation and Parasitism which are based on studies carried out by researchers on the inter-relationship between creatures.

It is noteworthy to mention that the general concept of modeling has been presented in the treatises of Meyer [3], Cushing [4], Kapur [5, 6], Srinivas [7] who studied competitive ecosystem of two species and three species with limited and unlimited resources. On the other hand Laxminarayan and Pattabhi Ramacharyulu [8] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. At the same time, Archana Reddy [9] and Bhaskara Rama Sharma [10] investigated diverse problems related to two species competitive systems with time delay by employing analytical and numerical techniques, Phani Kumar [11] studied some mathematical models of ecological commensalism and Ravindra Reddy [12] discussed on the stability of two mutually interacting species with mortality rate for the second species. Further Srilatha [13, 14] and Shiva

Reddy [15] studied stability analysis of three and four species. Hari Prasad and Pattabhi Ramacharyulu [16-21] discussed on the stability of a three and four species syn-ecosystems. The present authors [22-25] investigated the stability of three species and four species with stage structure, optimal harvesting policy and stochasticity. Hari prasad [26], Kar [27], Carletti [28] and Nisbet [29] inspired us to do the present investigation on the analytical and numerical approach of a emblematical four species syn-ecosystem.

The paper is organized as follows: Section 1 is a major part that deals with the introduction we have just seen. It records our research and previous research that has been carried out on the syn-ecosystems. Section 2 describes the mathematical model (2.1)-(2.4). Further we analyze the stability of deterministic model (3.1)-(3.4) in section 3 that consists of four sub-sections on steady states, local stability and global stability. In section 4, we compute analytical estimates of the population variances of the model (2.1)-(2.4). Physical significances of population variances are given in section 5. The given computer simulations in section 6 helps us to validate the theoretical results. Section 7 is based on the conclusions and section 8 highlights the future scope of the present work.

II. MATHEMATICAL MODEL

In this present paper, we assume the presence of randomly fluctuating driving forces on the growth of the species $S_i, i = 1,2,3,4$ at time 't' of a conventional syn-eco system. The table (2.1) exhibits some of the real examples of the present syn-eco system. The figure (2.1) represents the system where four species are living together with the following suppositions: (i) The system comprises of a prey (S_1), predator (S_2) two hosts S_3 and S_4 (ii) S_1 is prey of S_2 (ii) S_1 is commensal of S_3 (iii) S_2 is predator of S_1 (iv) S_2 is commensal of S_4 (v) S_3 is host of S_1 and (vi) S_4 is host of S_2 which results the following stochastic system with 'additive noise'. Let $x(t)$, $y(t)$, $z(t)$ and $w(t)$ be the population densities of species S_1, S_2, S_3 and S_4 respectively at time instant 't'. Let a_1, a_2, a_3 and a_4 be the natural growth rates of species S_1, S_2, S_3 and S_4 respectively. Keeping these in view and following [26-29], the dynamics of the stochastic system may be governed by the following nonlinear differential equations:

$$\frac{dx}{dt} = a_1 x - a_{11}x^2 - a_{12}xy + a_{13}xz - q_1E_1x + \beta_1\psi_1(t) \quad (2.1)$$

$$\frac{dy}{dt} = a_2 y - a_{22}y^2 + a_{21}xy + a_{24}yw - q_2E_2y + \beta_2\psi_2(t) \quad (2.2)$$

$$\frac{dz}{dt} = a_3 z - a_{33}z^2 + \beta_3\psi_3(t) \quad (2.3)$$

$$\frac{dw}{dt} = a_4 w - a_{44}w^2 + \beta_4\psi_4(t) \quad (2.4)$$

In the above model a_{11}, a_{22}, a_{33} and a_{44} are self inhibition coefficients of species S_1, S_2, S_3 and S_4 respectively. a_{12} is the interaction coefficient of S_1 due to S_2, a_{21} is the interaction coefficient of S_2 due to S_1, a_{13} is coefficient of commensal for S_1 due to the host S_3, a_{24} is the coefficient of commensal for S_2 due to the host S_4, K_1, K_2, K_3 and K_4 are the carrying capacities of species S_1, S_2, S_3 and S_4 respectively, where $K_1 = a_1/a_{11}$; $K_2 = a_2/a_{22}$; $K_3 = a_3/a_{33}$, $K_4 = a_4/a_{44}$. q_1, q_2 are the catchability coefficients of species S_1, S_2 respectively. E_1, E_2 are the efforts applied to harvest the species S_1, S_2 respectively. β_1, β_2, β_3 and β_4 are real constants and $\psi(t) = [\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t)]$ is a four dimensional Gaussian white noise process agreeable

$$E\left[\psi_i\left(t\right)\right]=0;\ i=1,2,3,4 \quad (2.5)$$

$$E\left[\psi_i\left(t\right)\psi_j\left(t'\right)\right]=\delta_{ij}\delta\left(t-t'\right);\ i=j=1,2,3,4 \quad (2.6)$$

where δ_{ij} and δ are Kronecker and Dirac delta functions respectively. In addition to the variables x, y, z, w the model parameters a_1, a_2, a_3, a_4, a_{11}, a_{22}, a_{33}, a_{44}, a_{12}, a_{21}, a_{13}, a_{24} are alleged to be non negative constants.

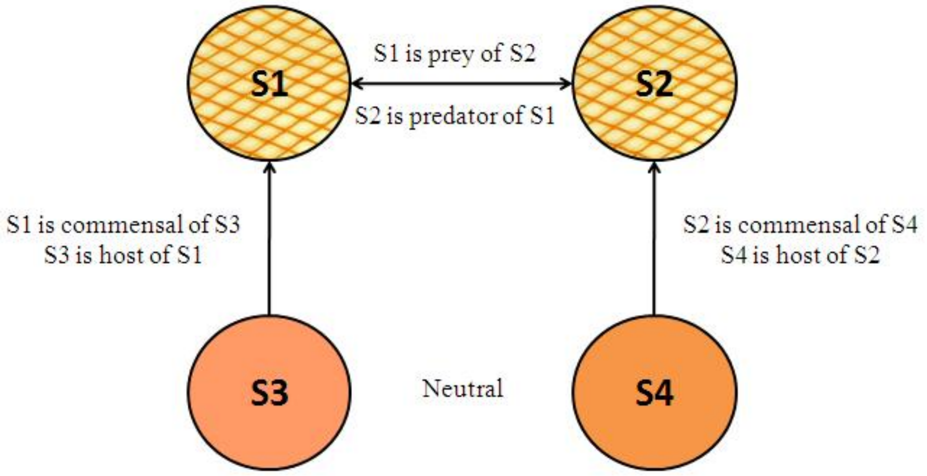

Figure 2.1: represents schematic idea of the four species living together

Table 2.1 represents some real examples of the system (2.1)-(2.4)

Example	S1	S2	S3	S4
1	Beetles	Small fish	phytoplankton	Zooplankton
2	Small bird	Man	Cow	Dog
3	Bugs	Insects	Rabin bird	Squirrel
4	Rabbit	Wolf	Bushes	Shelter-tree
5	Goat	Tiger	E-coli	Soil bacteria

III. STABILITY ANALYSIS OF DETERMINISTIC MODEL

In the absence of randomly fluctuating driving forces on the growth of the species, the model system (2.1)-(2.4) reduces to

$$\frac{dx}{dt}=x\left[\left(a_1-q_1E_1\right)-\left(a_{11}x+a_{12}y-a_{13}z\right)\right] \quad (3.1)$$

$$\frac{dy}{dt}=y\left[\left(a_2-q_2E_2\right)-\left(a_{22}y-a_{21}x-a_{24}w\right)\right] \quad (3.2)$$

$$\frac{dz}{dt}=z\left(a_3-a_{33}z\right) \quad (3.3)$$

$$\frac{dw}{dt}=w\left(a_4-a_{44}w\right) \quad (3.4)$$

Throughout our study let us suppose that

$$a_1-q_1E_1>0 \text{ and } a_2-q_2E_2>0 \quad (3.5)$$

3.1 Steady States

In this section, we present the basic outcomes on the nonnegative equilibriums of the model (3.1)-(3.4) namely $L_0(0,0,0)$, $L_1(\bar{x},\bar{y},0,0)$, $L_2(x^{\phi},y^{\phi},z^{\phi},0)$ and $L_3(x^*,y^*,z^*,w^*)$ which are attained by solving $\dot{x}=\dot{y}=\dot{z}=\dot{w}=0$

Case (i): $L_0(0,0,0)$: The population is extinct and this always exists.

Case (ii): $L_1(\bar{x},\bar{y},0,0)$: Here $\bar{x}$ and $\bar{y}$ are positive solutions of $\dot{x}=0$ and $\dot{y}=0$. We get

$$\bar{y}=\frac{1}{\left(a_{12}a_{21}+a_{22}a_{11}\right)}\left[a_{21}\left(a_1-q_1E_1\right)+a_{11}\left(a_2-q_2E_2\right)\right] \tag{3.1.1}$$

$$\bar{x}=\frac{1}{\left(a_{12}a_{21}+a_{22}a_{11}\right)}\left[a_{22}\left(a_1-q_1E_1\right)-a_{12}\left(a_2-q_2E_2\right)\right] \tag{3.1.2}$$

For $\bar{x}$ to be positive, we must have $\dfrac{a_{22}}{a_{12}}>\dfrac{a_2-q_2E_2}{a_1-q_1E_1}$ (3.1.3)

Case (iii): $L_2(x^{\phi},y^{\phi},z^{\phi},0)$: Here x^{ϕ}, y^{ϕ} and z^{ϕ} are positive solutions of $\dot{x}=0$ $\dot{y}=0$ and $\dot{z}=0$. We get

$$x^{\phi}=\frac{1}{a_{33}\left(a_{12}a_{21}+a_{22}a_{11}\right)}\left[a_{22}a_3a_{13}+a_{33}a_{22}\left(a_1-q_1E_1\right)-a_{33}a_{12}\left(a_2-q_2E_2\right)\right] \tag{3.1.4}$$

$$y^{\phi}=\frac{1}{a_{33}\left(a_{12}a_{21}+a_{22}a_{11}\right)}\left[a_{21}a_3a_{13}+a_{33}a_{21}\left(a_1-q_1E_1\right)+a_{33}a_{11}\left(a_2-q_2E_2\right)\right] \tag{3.1.5}$$

$$z^{\phi}=\frac{a_3}{a_{33}} \tag{3.1.6}$$

For x^{ϕ} to be positive, we must have $\dfrac{a_{22}}{a_{12}}>\dfrac{a_2-q_2E_2}{a_1-q_1E_1}$ (3.1.7)

Case (iv): $L_3(x^*,y^*,z^*,w^*)$ (The interior equilibrium): Here x^*, y^*, z^* and w^* are positive solutions of $\dot{x}=0$, $\dot{y}=0$, $\dot{z}=0$ and $\dot{w}=0$.

We get

$$z^*=\frac{a_3}{a_{33}} \tag{3.1.8}$$

$$w^*=\frac{a_4}{a_{44}} \tag{3.1.9}$$

$$x^*=\frac{1}{a_{33}a_{44}\left(a_{12}a_{21}+a_{22}a_{11}\right)}\left[\begin{array}{r}a_{44}a_{13}a_3a_{22}+a_{33}a_{44}a_{22}\left(a_1-q_1E_1\right)-a_{12}a_{33}a_{24}a_4\\-a_{12}a_{33}a_{44}\left(a_2-q_2E_2\right)\end{array}\right] \tag{3.1.10}$$

$$y^*=\frac{1}{a_{33}a_{44}\left(a_{12}a_{21}+a_{22}a_{11}\right)}\left[\begin{array}{r}a_{44}a_{13}a_3a_{21}+a_{11}a_{33}a_{24}a_4+a_{33}a_{44}a_{21}\left(a_1-q_1E_1\right)\\+a_{33}a_{44}a_{11}\left(a_2-q_2E_2\right)\end{array}\right] \tag{3.1.11}$$

For x^* to be positive we must have the following:

$$\frac{a_3}{a_4}>\frac{a_{12}a_{33}a_{24}}{a_{44}a_{13}a_{22}} \tag{3.1.12}$$

$$\frac{a_{22}}{a_{12}}>\frac{a_2-q_2E_2}{a_1-q_1E_1} \tag{3.1.13}$$

The similar work has been carried out by ChaoLiu [30].

3.2. Local Stability

We now analyze the local stability of the interior steady state [31]. The Variational matrix of the system (3.1)-(3.4) at $L_3(x^*,y^*,z^*,w^*)$ is

$$J = \begin{pmatrix} -a_{11}x^* & -a_{12}x^* & a_{13}x^* & 0 \\ a_{21}y^* & -a_{22}y^* & 0 & a_{24}y^* \\ 0 & 0 & -a_3 & 0 \\ 0 & 0 & 0 & -a_4 \end{pmatrix} \quad (3.2.1)$$

The characteristic equation of (3.2.1) is in the form of $\omega^4 + A\omega^3 + B\omega^2 + C\omega + D = 0$ (3.2.2)

where $A = a_{11}x^* + a_3 + a_4 > 0$, $B = a_{11}a_{22}x^*y^* + a_{11}a_3x^*y^* + a_{11}a_4x^*y^* + a_3a_4 > 0$,

$C = a_{11}a_{22}a_4x^*y^* + a_{11}a_{22}a_3x^*y^* + a_{21}a_3x^*y^* + a_{21}a_4x^*y^* > 0$,

$D = a_{11}a_{22}a_3a_4x^*y^* + a_{21}a_3a_4x^*y^* > 0$.

The system is locally asymptotically stable if all the eigen values of the above characteristic equation have negative real parts. By Routh-Hurwitz criteria, it follows that all eigen values of (3.2.2) have negative real parts if and only if $A > 0, C > 0$, $D > 0$, $C(AB - C) > A^2D$, $D(ABC - A^2D - C^2) > 0$. Hence $L_3(x^*, y^*, z^*, w^*)$ is locally asymptotically stable.

3.3. Global Stability

Now we discuss the global stability [32] of the equilibrium points $L_1(\bar{x}, \bar{y}, 0, 0)$ and $L_3(x^*, y^*, z^*, w^*)$ of the system (3.1)-(3.4).

Theorem (3.3.1): The Equilibrium point $L_1(\bar{x}, \bar{y}, 0, 0)$ is globally asymptotically stable.

Proof: let us consider the following Lyapunov function

$$V(x, y) = \left(x - \bar{x} - \bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right) + l_1\left(y - \bar{y} - \bar{y}\ln\left(\frac{y}{\bar{y}}\right)\right)$$

Differentiating V w.r.to 't' we get

$$\frac{dV}{dt} = \left(\frac{x - \bar{x}}{x}\right)\frac{dx}{dt} + l_1\left(\frac{y - \bar{y}}{y}\right)\frac{dy}{dt};$$

$$\frac{dV}{dt} = -a_{11}(x - \bar{x})^2 - l_1a_{22}(y - \bar{y})^2 + (l_1a_{21} - a_{12})(x - \bar{x})(y - \bar{y})$$

By choosing $l_1 = a_{12}/a_{21}$, $dV/dt = -\left[a_{11}(x - \bar{x})^2 - \frac{a_{22}a_{12}}{a_{21}}(y - \bar{y})^2\right] < 0$

Hence the equilibrium point $L_1(\bar{x}, \bar{y}, 0, 0)$ is globally asymptotically stable.

Theorem (3.3.2): The interior equilibrium point $L_3(x^*, y^*, z^*, w^*)$ is globally asymptotically stable if $4a_{21}a_{22} > a_{12}a_{24}^2$ and $4a_{11} > a_{13}^2$.

Proof: To find the condition for global stability at $L_3(x^*, y^*, z^*, w^*)$, we construct the Lyapunov function

$$V(x, y, z, w) = \left[(x - x^*) - x^*\ln(x/x^*)\right] + l_1\left[(y - y^*) - y^*\ln(y/y^*)\right] + l_2\left[(z - z^*) - z^*\ln(z/z^*)\right]$$

$$+ l_3\left[(w - w^*) - w^*\ln(w/w^*)\right]$$

where l_1, l_2 and l_3 are positive constants.

$$(dV/dt) = \left[(x - x^*)/x\right](dx/dt) + l_1\left[(y - y^*)/y\right](dy/dt) + l_2\left[(z - z^*)/z\right](dz/dt)$$
$$+ l_3\left[(w - w^*)/w\right](dw/dt);$$

$$(dV/dt) = (x - x^*)\left[-a_{11}(x - x^*) - a_{12}(y - y^*) + a_{13}(z - z^*)\right]$$

$$+l_1\left(y-y^*\right)\left[-a_{22}\left(y-y^*\right)+a_{21}\left(x-x^*\right)+a_{24}\left(w-w^*\right)\right]$$
$$+l_2\left(z-z^*\right)\left[-a_{33}\left(z-z^*\right)\right]+l_3\left(w-w^*\right)\left[-a_{44}\left(w-w^*\right)\right];$$

By choosing $l_1=\frac{a_{12}}{a_{21}}; l_2=\frac{1}{a_{33}}; l_3=\frac{1}{a_{44}}$

$$(dV/dt)=-\begin{bmatrix} a_{11}\left(x-x^*\right)^2-a_{13}\left(x-x^*\right)\left(z-z^*\right)+\left(a_{12}a_{22}/a_{21}\right)\left(y-y^*\right)^2 \\ -\left(a_{12}a_{24}/a_{21}\right)\left(y-y^*\right)\left(w-w^*\right)+\left(z-z^*\right)^2+\left(w-w^*\right)^2 \end{bmatrix}$$
$$=-X^TAX$$

where $X=\begin{bmatrix} x-x^* \\ y-y^* \\ z-z^* \\ w-w^* \end{bmatrix}; A=\begin{bmatrix} a_{11} & 0 & \frac{-a_{13}}{2} & 0 \\ 0 & \frac{a_{12}a_{22}}{a_{21}} & 0 & -\frac{a_{12}a_{24}}{2a_{21}} \\ \frac{-a_{13}}{2} & 0 & 1 & 0 \\ 0 & -\frac{a_{12}a_{24}}{2a_{21}} & 0 & 1 \end{bmatrix}$

The system is globally stable if the derivative of Lyapunav's function V is negative definite, that is if the matrix A is positive definite, that is if the principal minors of A (say) $M_i, i=1,2,3,4$ are positive. The principle minors are positive if $4a_{21}a_{22}>a_{12}a_{24}^2$ and $4a_{11}>a_{13}^2$. Hence the system is globally stable in the above parametric domain.

3.4. Bionomic Equilibrium

It is the combination of biological balance and economic balance. In section (3.1), we have conversed about the biological balance which is given by $\dot{x}=\dot{y}=\dot{z}=\dot{w}=0$. When the total profit obtained by selling the yielded biomass equals the total cost utilized in yielding it, then we say that the bionomic balance achieved. Let c_1 be the constant harvesting cost of species S_1 per unit effort and c_2 be the constant harvesting cost of species S_2 per unit effort. Let p_1 be the constant price of species S_1 per unit biomass and p_2 be the constant price of species S_2 per unit biomass. The revenue at any time is given by

$$A\left(x,y,z,w,E_1,E_2\right)=\left(p_1q_1x-c_1\right)E_1+\left(p_2q_2z-c_2\right)E_2 \quad (3.4.1)$$

Now if $c_1>p_1q_1x$ and $c_2>p_2q_2y$, then the economic rent obtained from the fishery becomes negative and the fishery will be closed. Hence for the existence of bionomic equilibrium, it is assumed that $c_1<p_1q_1x$ and $c_2<p_2q_2y$. (3.4.2)

The bionomic equilibrium $\left((x)_\infty,(y)_\infty,(z)_\infty,(w)_\infty,(E_1)_\infty,(E_2)_\infty\right)$ is the positive solution of

$$\dot{x}=\dot{y}=\dot{z}=\dot{w}=A=0. \quad (3.4.3)$$

By solving (3.4.3) we get,

$$(x)_\infty=c_1/(p_1q_1); \quad (3.4.4)$$
$$(y)_\infty=c_2/(p_2q_2); \quad (3.4.5)$$
$$(z)_\infty=a_3/a_{33}; \quad (3.4.6)$$
$$(w)_\infty=a_4/a_{44}; \quad (3.4.7)$$
$$(E_1)_\infty=(1/q_1)\left[a_1-(a_{11}c_1)/(p_1q_1)-(a_{12}c_2)/(p_2q_2)+(a_{13}a_3)/a_{33}\right] \quad (3.4.8)$$
$$\left(E_2\right)_\infty=(1/q_2)\left[a_2-(a_{22}c_2)/(p_2q_2)+(a_{21}c_1)/(p_1q_1)+(a_{24}a_4)/a_{44}\right] \quad (3.4.9)$$

$$(E_1)_\infty > 0 \text{ when} \left[a_1 + (a_{13}a_3)/a_{33}\right] > \left[(a_{11}c_1)/(p_1q_1) + (a_{12}c_2)/(p_2q_2)\right]. \quad (3.4.10)$$

$$(E_2)_\infty > 0 \text{ when} \left[a_2 + (a_{21}c_1)/(p_1q_1) + (a_{24}a_4)/a_{44}\right] > \left[(a_{22}c_2)/(p_2q_2)\right]. \quad (3.4.11)$$

If $(E_1) > (E_1)_\infty$ and $(E_2) > (E_2)_\infty$, then the total cost utilized in harvesting the species population would exceed the total revenues obtained from the ecological system. Hence some people would be in loss and naturally they would withdraw their participation from the system. Hence $(E_1) > (E_1)_\infty$ and $(E_2) > (E_2)_\infty$ cannot be maintained indefinitely.

If $(E_1) < (E_1)_\infty$ and $(E_2) < (E_2)_\infty$, then the ecological system is more profitable, and hence in an open access system, it would attract more and more people. This will have an increasing effect on the yielding effort. Hence $(E_1) < (E_1)_\infty$ and $(E_2) < (E_2)_\infty$ cannot be continued indefinitely.

IV. STOCHASTIC ANALYSIS

In this section, we compute the population intensities of fluctuations (variances) of the system (2.1)-(2.4) around the positive equilibrium $L_3(x^*, y^*, z^*, w^*)$ due to noise, according to the method introduced by Nisbet and Gurney [29] in 1982. The method was successfully applied by Prasenjit Das [33] and M.N. Srinivas [34]. Now we assume the presence of randomly fluctuating driving forces on the deterministic growth of the species $S_i, i = 1, 2, 3$ of the system (3.1)-(3.4) at time 't' which results in the stochastic system (2.1)-(2.4) with 'additive white noise' process satisfying (2.5) and (2.6). Let us consider the perturbation technique as follows:

Let $x(t) = u_1(t) + S^*; y(t) = u_2(t) + P^*; z(t) = u_3(t) + T^*; w(t) = u_4(t) + U^*;$ (4.1)

$$\frac{dx}{dt} = \frac{du_1(t)}{dt}; \frac{dy}{dt} = \frac{du_2(t)}{dt}; \frac{dz}{dt} = \frac{du_3(t)}{dt}; \frac{dw}{dt} = \frac{du_4(t)}{dt}; \quad (4.2)$$

Using (4.1) and (4.2) in (2.1)-(2.4), we identify the respective linear system as

$$\frac{du_1(t)}{dt} = -a_{11}u_1(t)S^* - a_{12}u_2(t)S^* + a_{13}u_3(t)S^* + \beta_1\psi_1(t) \quad (4.3)$$

$$\frac{du_2(t)}{dt} = -a_{22}u_2(t)P^* + a_{21}u_1(t)P^* + a_{24}u_4(t)P^* + \beta_2\psi_2(t) \quad (4.4)$$

$$\frac{du_3(t)}{dt} = -a_{33}u_3(t)T^* + \beta_3\psi_3(t) \quad (4.5)$$

$$\frac{du_4(t)}{dt} = -a_{44}u_4(t)U^* + \beta_4\psi_4(t) \quad (4.6)$$

Using Fourier transform methods on the linear system (4.3) - (4.6), we get

$$\beta_1\tilde{\psi}_1(\omega) = (i\omega + a_{11}S^*)\tilde{u}_1(\omega) + a_{12}S^*\tilde{u}_2(\omega) - a_{13}S^*\tilde{u}_3(\omega) \quad (4.7)$$

$$\beta_2\tilde{\psi}_2(\omega) = -a_{21}P^*u_1(\omega) + (i\omega + a_{22}P^*)\tilde{u}_2(\omega) - a_{24}P^*\tilde{u}_4(\omega) \quad (4.8)$$

$$\beta_3\tilde{\psi}_3(\omega) = (i\omega + a_{33}T^*)\tilde{u}_3(\omega) \quad (4.9)$$

$$\beta_4\tilde{\psi}_4(\omega) = (i\omega + a_{44}U^*)\tilde{u}_4(\omega) \quad (4.10)$$

The above system (4.7) - (4.10) can be represented in the matrix form as

$$A(\omega)\tilde{u}(\omega) = \tilde{\psi}(\omega) \quad (4.11)$$

where $$A(\omega) = \begin{pmatrix} A_{11}(\omega) & A_{12}(\omega) & A_{13}(\omega) & A_{14}(\omega) \\ A_{21}(\omega) & A_{22}(\omega) & A_{23}(\omega) & A_{24}(\omega) \\ A_{31}(\omega) & A_{32}(\omega) & A_{33}(\omega) & A_{34}(\omega) \\ A_{41}(\omega) & A_{42}(\omega) & A_{43}(\omega) & A_{44}(\omega) \end{pmatrix}; \tilde{u}(\omega) = \begin{bmatrix} \tilde{u}_1(\omega) \\ \tilde{u}_2(\omega) \\ \tilde{u}_3(\omega) \\ \tilde{u}_4(\omega) \end{bmatrix}; \tilde{\psi}(\omega) = \begin{bmatrix} \beta_1\tilde{\psi}_1(\omega) \\ \beta_2\tilde{\psi}_2(\omega) \\ \beta_3\tilde{\psi}_3(\omega) \\ \beta_4\tilde{\psi}_4(\omega) \end{bmatrix};$$

$$
\begin{aligned}
&A_{11}(\omega)=(i\omega+a_{11}S^{*});\ A_{12}(\omega)=a_{12}S^{*};\ A_{13}(\omega)=-a_{13}S^{*};\ A_{14}(\omega)=0;\\
&A_{21}(\omega)=-a_{21}P^{*};\ A_{22}(\omega)=(i\omega+a_{22}P^{*});\ A_{23}(\omega)=0;\ A_{24}(\omega)=-a_{24}P^{*};\\
&A_{31}(\omega)=0;\ A_{32}(\omega)=0;\ A_{33}(\omega)=(i\omega+a_{33}T^{*});\ A_{34}(\omega)=0;\\
&A_{41}(\omega)=0; A_{42}(\omega)=0; A_{43}(\omega)=0; A_{44}(\omega)=(i\omega+a_{44}U^{*});
\end{aligned} \tag{4.12}
$$

Equation (4.11) can also be written as $\tilde{u}(\omega)=\left[A(\omega)\right]^{-1}\tilde{\psi}(\omega)$

$$\text{Let}\left[A(\omega)\right]^{-1}=B(\omega)\text{ then } B(\omega)=\frac{Adj\,A(\omega)}{\left|A(\omega)\right|}\text{ and } \tilde{u}(\omega)=B(\omega)\tilde{\psi}(\omega) \tag{4.13}$$

where $\left|A(\omega)\right|=R(\omega)+iI(\omega)$

$$
\begin{aligned}
R(\omega)=\ &\omega^{4}-\omega^{2}a_{33}a_{44}T^{*}U^{*}-\omega^{2}a_{22}a_{44}P^{*}U^{*}-\omega^{2}a_{22}a_{33}P^{*}T^{*}-\omega^{2}a_{11}a_{44}S^{*}U^{*}\\
&-\omega^{2}a_{11}a_{33}S^{*}T^{*}-\omega^{2}a_{11}a_{22}S^{*}P^{*}-\omega^{2}a_{12}a_{21}S^{*}P^{*}+a_{11}a_{22}a_{33}a_{44}S^{*}P^{*}T^{*}U^{*}\\
&+a_{12}a_{21}a_{33}a_{44}S^{*}P^{*}T^{*}U^{*}
\end{aligned} \tag{4.14}
$$

$$
\begin{aligned}
I(\omega)=\ &-\omega^{3}a_{11}S^{*}-\omega^{3}a_{22}P^{*}-\omega^{3}a_{33}T^{*}-\omega^{3}a_{44}U^{*}+\omega a_{22}a_{33}a_{44}P^{*}T^{*}U^{*}\\
&+\omega a_{11}a_{33}a_{44}S^{*}T^{*}U^{*}+\omega a_{11}a_{22}a_{44}S^{*}P^{*}U^{*}+\omega a_{11}a_{22}a_{33}S^{*}P^{*}T^{*}\\
&+\omega a_{12}a_{21}a_{33}S^{*}P^{*}T^{*}+\omega a_{12}a_{21}a_{44}S^{*}P^{*}U^{*}
\end{aligned}
$$

We now depict some of the necessary preliminaries of the random population function. If the function $Y(t)$ has a zero mean value, then the fluctuation intensity (variance) of its components in the frequency interval $\left[\omega,\omega+d\omega\right]$ is $S_Y(\omega)d\omega$, where $S_Y(\omega)$ is spectral density of Y and is defined as

$$S_Y(\omega)=\lim_{\tilde{T}\to\infty}\frac{\left|\tilde{Y}(\omega)\right|^{2}}{\tilde{T}} \tag{4.15}$$

If Y has a zero mean value, the inverse transform of $S_Y(\omega)$ is the auto covariance function

$$C_Y(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_Y(\omega)e^{i\omega\tau}d\omega \tag{4.16}$$

The corresponding variance of fluctuations in $Y(t)$ is given by

$$\sigma_Y^{\ 2}=C_Y(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_Y(\omega)d\omega \tag{4.17}$$

and the auto correlation function is the normalized auto covariance $P_Y(\tau)=\dfrac{C_Y(\tau)}{C_Y(0)}$ (4.18)

For a Gaussian white noise process, it is

$$
\begin{aligned}
S_{\psi_i\psi_j}(\omega)&=\lim_{\hat{T}\to+\infty}\frac{E\left[\tilde{\psi}_i(\omega)\tilde{\psi}_j(\omega)\right]}{\hat{T}}\\
&=\lim_{\hat{T}\to+\infty}\frac{1}{\hat{T}}\int_{-\frac{\hat{T}}{2}}^{\frac{\hat{T}}{2}}\int_{-\frac{\hat{T}}{2}}^{\frac{\hat{T}}{2}}E\left[\tilde{\psi}_i(t)\tilde{\psi}_j(t')\right]e^{-i\omega(t-t')}dt\,dt'=\delta_{ij}
\end{aligned} \tag{4.19}
$$

From (4.13), we have $\tilde{u}_i(\omega)=\sum_{j=1}^{4}B_{ij}(\omega)\tilde{\psi}_j(\omega)$, $i=1,2,3,4$ (4.20)

From (4.14) we have $S_{u_i}(\omega)=\sum_{j=1}^{4}\beta_j\left|B_{ij}(\omega)\right|^{2}$, $i=1,2,3,4$ (4.21)

Hence by (4.17) and (4.21), the intensities of fluctuations of the variables u_i, $i=1,2,3,4$ are given by $\sigma_{u_i}{}^2 = \frac{1}{2\pi}\sum_{j=1}^{4}\int_{-\infty}^{\infty}\beta_j\left|B_{ij}(\omega)\right|^2 d\omega;\ i=1,2,3,4$ (4.22)

That is, the variances of u_i, $i=1,2,3,4$ are obtained as

$$\sigma_{u_1}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\beta_1\left|B_{11}(\omega)\right|^2 d\omega + \int_{-\infty}^{\infty}\beta_2\left|B_{12}(\omega)\right|^2 d\omega + \int_{-\infty}^{\infty}\beta_3\left|B_{13}(\omega)\right|^2 d\omega + \int_{-\infty}^{\infty}\beta_4\left|B_{14}(\omega)\right|^2 d\omega\right\};$$

$$\sigma_{u_2}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\beta_1\left|B_{21}(\omega)\right|^2 d\omega + \int_{-\infty}^{\infty}\beta_2\left|B_{22}(\omega)\right|^2 d\omega + \int_{-\infty}^{\infty}\beta_3\left|B_{23}(\omega)\right|^2 d\omega + \int_{-\infty}^{\infty}\beta_4\left|B_{24}(\omega)\right|^2 d\omega\right\};$$

$$\sigma_{u_3}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\beta_1\left|B_{31}(\omega)\right|^2 d\omega + \int_{-\infty}^{\infty}\beta_2\left|B_{32}(\omega)\right|^2 d\omega + \int_{-\infty}^{\infty}\beta_3\left|B_{33}(\omega)\right|^2 d\omega + \int_{-\infty}^{\infty}\beta_4\left|B_{34}(\omega)\right|^2 d\omega\right\};$$

$$\sigma_{u_4}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\beta_1\left|B_{41}(\omega)\right|^2 d\omega + \int_{-\infty}^{\infty}\beta_2\left|B_{42}(\omega)\right|^2 d\omega + \int_{-\infty}^{\infty}\beta_3\left|B_{43}(\omega)\right|^2 d\omega + \int_{-\infty}^{\infty}\beta_4\left|B_{44}(\omega)\right|^2 d\omega\right\}$$

(4.23)

where $B_{mn}(\omega) = \frac{X_{mn}+iY_{mn}}{R(\omega)+iI(\omega)};\ m,n=1,2,3,4$

$$X_{11} = -\omega^2 a_{22}P^* - \omega^2 a_{33}T^* - \omega^2 a_{44}U^* + \omega a_{22}a_{33}a_{44}P^*T^*U^*;$$
$$Y_{11} = -\omega^3 + \omega a_{22}a_{33}P^*T^* + \omega a_{33}a_{44}T^*U^* + \omega a_{22}a_{44}P^*U^*;\ X_{12} = \omega^2 a_{12}S^*;$$
$$Y_{12} = -\omega a_{12}a_{33}S^*T^* - \omega a_{12}a_{44}S^*U^*;\ X_{13} = -\omega^2 a_{13}S^* + a_{13}a_{22}a_{44}S^*P^*U^*;$$
$$Y_{13} = \omega a_{13}a_{22}S^*P^* + \omega a_{13}a_{44}S^*U^*;\ X_{14} = -a_{12}a_{24}a_{33}S^*P^*T^*;\ Y_{14} = -\omega a_{12}a_{24}S^*P^*;$$
$$X_{21} = -\omega^2 a_{21}P^* + a_{21}a_{33}a_{44}P^*T^*U^*;\ Y_{21} = \omega a_{21}a_{33}P^*T^* + \omega a_{21}a_{44}P^*U^*;$$
$$X_{22} = -\omega^2 a_{11}S^* - \omega^2 a_{33}T^* - \omega^2 a_{44}U^* + a_{11}a_{33}a_{44}S^*T^*U^*;$$
$$Y_{22} = -\omega^3 + \omega a_{33}a_{44}T^*U^* + \omega a_{11}a_{44}S^*U^* + \omega a_{11}a_{33}S^*T^*;\ X_{23} = a_{13}a_{21}a_{44}S^*P^*U^*;$$
$$Y_{23} = \omega a_{13}a_{21}S^*P^*;\ Y_{24} = \omega a_{11}a_{24}S^*P^* + \omega a_{24}a_{33}P^*T^*;\ \ Y_{31} = 0;\ X_{32} = 0;\ Y_{32} = 0;$$
$$X_{33} = -\omega^2 a_{11}S^* - \omega^2 a_{22}P^* - \omega^2 a_{44}U^* + a_{11}a_{22}a_{44}S^*P^*U^* + a_{12}a_{21}a_{44}S^*P^*U^*;$$
$$Y_{33} = -\omega^3 + \omega a_{22}a_{44}P^*U^* + \omega a_{11}a_{44}S^*U^* + \omega a_{11}a_{22}S^*P^* + \omega a_{12}a_{21}S^*P^*;$$
$$X_{34} = 0;\ Y_{34} = 0;\ X_{41} = 0;\ Y_{41} = 0;\ X_{42} = 0;\ Y_{42} = 0;\ X_{43} = 0;\ Y_{43} = 0;$$
$$X_{44} = -\omega^2 a_{11}S^* - \omega^2 a_{22}P^* - \omega^2 a_{33}T^* + a_{11}a_{22}a_{33}S^*P^*T^* + a_{12}a_{21}a_{33}S^*P^*T^*;$$
$$Y_{44} = -\omega^3 + \omega a_{22}a_{33}P^*T^* + \omega a_{11}a_{33}S^*T^* + \omega a_{11}a_{22}S^*P^* + \omega a_{12}a_{21}S^*P^*;$$

Thus (4.23) becomes

$$\sigma_{u_1}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}\left[\begin{array}{l}\beta_1\left(X_{11}{}^2+Y_{11}{}^2\right)+\beta_2\left(X_{12}{}^2+Y_{12}{}^2\right)\\ \qquad +\beta_3\left(X_{13}{}^2+Y_{13}{}^2\right)+\beta_4\left(X_{14}{}^2+Y_{14}{}^2\right)\end{array}\right]d\omega\right\};$$

$$\sigma_{u_2}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}\left[\begin{array}{l}\beta_1\left(X_{21}{}^2+Y_{21}{}^2\right)+\beta_2\left(X_{22}{}^2+Y_{22}{}^2\right)\\ \qquad +\beta_3\left(X_{23}{}^2+Y_{23}{}^2\right)+\beta_4\left(X_{24}{}^2+Y_{24}{}^2\right)\end{array}\right]d\omega\right\};$$

$$\sigma_{u_3}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}\left[\begin{array}{l}\beta_1\left(X_{31}{}^2+Y_{31}{}^2\right)+\beta_2\left(X_{32}{}^2+Y_{32}{}^2\right)\\ \qquad +\beta_3\left(X_{33}{}^2+Y_{33}{}^2\right)+\beta_4\left(X_{34}{}^2+Y_{34}{}^2\right)\end{array}\right]d\omega\right\};$$

$$\sigma_{u_4}{}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\frac{1}{R^2(\omega)+I^2(\omega)}\left[\begin{array}{l}\beta_1\left(X_{41}{}^2+Y_{41}{}^2\right)+\beta_2\left(X_{42}{}^2+Y_{42}{}^2\right)\\ \qquad +\beta_3\left(X_{43}{}^2+Y_{43}{}^2\right)+\beta_4\left(X_{44}{}^2+Y_{44}{}^2\right)\end{array}\right]d\omega\right\} \tag{4.24}$$

If we want know the behaviour of the system (2.1)-(2.4) with either $\beta_1 = 0$ or $\beta_2 = 0$ or $\beta_3 = 0$ or $\beta_4 = 0$, then the variances are :

If $\beta_1 = \beta_2 = \beta_3 = 0$, then $\sigma_{u_1}{}^2 = \frac{\beta_4}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{14}{}^2+Y_{14}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;\ \sigma_{u_2}{}^2 = \frac{\beta_4}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{24}{}^2+Y_{24}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;$

$$\sigma_{u_3}{}^2 = \frac{\beta_4}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{34}{}^2+Y_{34}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega = 0;\quad \sigma_{u_4}{}^2 = \frac{\beta_4}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{44}{}^2+Y_{44}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega.$$

If $\beta_1 = \beta_2 = \beta_4 = 0$, then $\sigma_{u_1}{}^2 = \frac{\beta_3}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{13}{}^2+Y_{13}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;\ \sigma_{u_2}{}^2 = \frac{\beta_3}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{23}{}^2+Y_{23}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;$

$$\sigma_{u_3}{}^2 = \frac{\beta_3}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{33}{}^2+Y_{33}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;\quad \sigma_{u_4}{}^2 = \frac{\beta_3}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{43}{}^2+Y_{43}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega = 0.$$

If $\beta_1 = \beta_3 = \beta_4 = 0$, then $\sigma_{u_1}{}^2 = \frac{\beta_2}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{12}{}^2+Y_{12}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;\ \sigma_{u_2}{}^2 = \frac{\beta_2}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{22}{}^2+Y_{22}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;$

$$\sigma_{u_3}{}^2 = \frac{\beta_2}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{32}{}^2+Y_{32}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega = 0;\quad \sigma_{u_4}{}^2 = \frac{\beta_2}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{42}{}^2+Y_{42}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega = 0.$$

If $\beta_2 = \beta_3 = \beta_4 = 0$, then $\sigma_{u_1}{}^2 = \frac{\beta_1}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{11}{}^2+Y_{11}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;\ \sigma_{u_2}{}^2 = \frac{\beta_1}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{21}{}^2+Y_{21}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega;$

$$\sigma_{u_2}{}^2 = \frac{\beta_1}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{31}{}^2+Y_{31}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega = 0;\quad \sigma_{u_4}{}^2 = \frac{\beta_1}{2\pi}\int_{-\infty}^{\infty}\frac{\left(X_{41}{}^2+Y_{41}{}^2\right)}{R^2(\omega)+I^2(\omega)}d\omega = 0.$$

V. PHYSICAL SIGNIFICANCE

Analytical evaluation of the integrals in (4.24) is difficult, but it can be evaluated numerically for a different set of values of parameters. The four variances in (4.24) represent the mean square fluctuations of the populations. These are fluctuations of the population from the mean values of the population. When the variances are less, we can say that the system is stable whereas when the variances are more, the system is unstable. In the computer simulation we can identify the parametric domain in which the system has stable equilibrium where population variances are small and also the parametric space in which the system has unstable equilibrium where the population variances are large.

VI. COMPUTER SIMULATION

For substantiation of our earlier discussed analytical results, we here would like to present some numerical replications with the help of MATLAB 7.3 software package.

Example (6.1):

$$a_1 = 3; a_{11} = 0.01; a_{12} = 0.45; a_{13} = 0.08; q_1 = 0.2; E_1 = 10, a_2 = 2; a_{22} = 0.5; a_{24} = 0.3; q_2 = 0.1$$
$$E_2 = 10; a_3 = 1.2; a_{33} = 0.2; a_4 = 1.5; a_{44} = 0.5$$

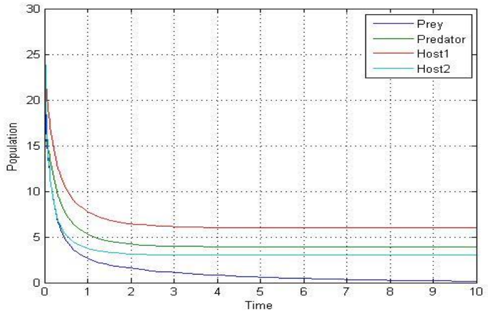

Figure 6.1(a)

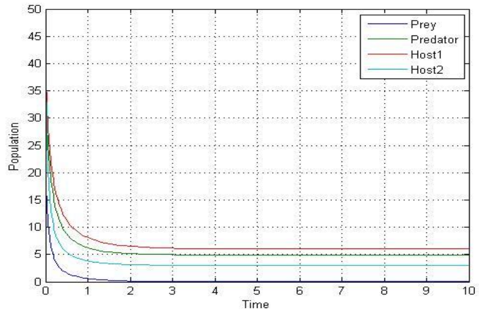

Figure 6.2 (b)

Figures 6.1(a) and 6.2 (b) show that the variation of population against time initially with $x = 20; y = 15; z = 25; w = 30$ and $x = 20; y = 30; z = 40; w = 50$ respectively.

Example (6.2):

$$a_1 = 3.5; a_{11} = 0.1; a_{12} = 0.5; a_{13} = 0.08; q_1 = 0.2; E_1 = 10; a_2 = 2.6; a_{22} = 0.5; a_{24} = 0.32$$
$$q_2 = 0.1; E_2 = 10; a_3 = 1.2; a_{33} = 0.2; a_4 = 1.5; a_{44} = 0.5$$

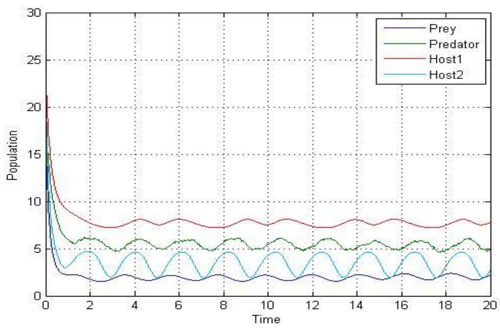

Figure 6.2 (a)

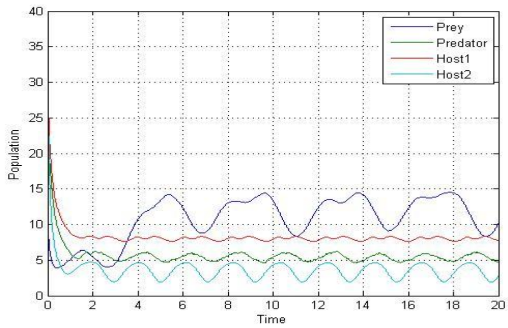

Figure 6.2 (b)

Figures 6.2(a) and 6.2(b) show that the variation of population against time initially with $x = 20; y = 15; z = 25; w = 30$ and $x = 10; y = 20; z = 30; w = 40$ respectively.

VII. CONCLUSION

In this paper, a model of a distinctive four species syn-ecosystem with stochastic term was invented. At first we have discussed the model without the stochastic term and examined the survival of equilibrium points as well as the local stability by utilizing Routh-Hurwitz criteria and the global stability using Lyapunov function. We found the bionomic equilibrium subjected to economic constraints. Later we added the stochastic term in the model and investigated the effect of environmental fluctuations around the positive equilibrium due to additive white noise. The population variances are computed and analyzed for stability using Matlab.

The analytical results and numerical simulation of deterministic four species system model suggest that the deterministic system is stable. The stable nature of the system is revealed in figures 6.1(a) & 6.1(b). Further for stochastic system, population variances have a great role to analyze the stability of the system. The conclusion is that the noise on the equation results in immense variances of oscillations around the equilibrium point which propose that our system is periodic with respect to a noisy atmosphere. Numerical replications reveal that the trajectories of the system oscillate arbitrarily with remarkable variance of amplitudes with the increasing value of the strength of noises initially but ultimately fluctuating which are viewed in figures 6.2(a) & 6.2(b). Hence we conclude that inclusion of stochastic perturbation create a significant change in intensity of the considered dynamical scheme due to change of responsive parameters which causes large environmental fluctuations.

VIII. FUTURE SCOPE

Following the same notation, we can incorporate time delay in the beneficial term due to predation of the predator population density equation as follows:

$$\frac{dy}{dt} = a_2 y - a_{22} y^2 + a_{21} x(t-\tau) y(t-\tau) + a_{24} yw - q_2 E_2 y + \beta_2 \psi_2(t)$$

It is very important to analyze the dynamical features of the model with time delay and to get an insight on the control of stability in the presence of time delay. The time delay analysis may produce interesting results that the increased time delay can increase the tendency for oscillatory behaviour and Hopf-bifurcation results.

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